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The Rayleigh problem for a wavy wall

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The problem of the flow generated in a viscous fluid by the impulsive motion of a wavy wall is treated as a perturbation about the known solution for a straight wall. It is shown that, while a unified treatment for high and low Reynolds numbers is possible in principle, the two limiting cases have to be treated separately in order to get results in closed form. It is also shown that a straightforward perturbation expansion in Reynolds number is not possible because the known solution is of exponential order in that parameter. At low Reynolds numbers the waviness of the wall quickly ceases to be of importance as the liquid is dragged along by the wall. At high Reynolds numbers on the other hand, the effects of viscosity are shown to be confined to a narrow layer close to the wall and the known potential solution emerges in time. The latter solution is a good illustration of the interaction between a viscous fluid field and its related inviscid field.

1. Introduction

The problem studied in this paper brings together two well-known examples used to illustrate the nature of fluid motion. The field generated by a flat plate impulsively moved in a viscous fluid is often used to demonstrate the generation of vorticity in a fluid by solid surfaces in motion relative to it. This problem, commonly (but apparently mistakenly) known as the Rayleigh problem, naturally leads to the study of viscous boundary layers. The wavy-wall problem, on the other hand, is a standard example in inviscid fluid dynamics used to illustrate the effect of boundary perturbations on the uniform motion of an inviscid fluid. The latter problem leads naturally to thin-aerofoil theory. In this paper we study the fluid motion generated by the impulsive motion of a wavy wall in a viscous incompressible fluid.

The problem is worthy of interest for a number of reasons. Problems in unsteady fluid dynamics such as this one show the development in time of a viscous field; they may therefore shed some light on the generation of turbulence. Second, the problem shows the interaction of a viscous field with an inviscid field. The results show the development in time of the inviscid far field in the high Reynolds number limit, i.e. the inviscid field and boundary layer emerge naturally from the analysis. Lastly, the analysis brings out clearly the care required in handling perturbation expansions in which terms of exponential order arise. The problem is formulated in § 2, the low Reynolds number limit is studied in § 3, the high

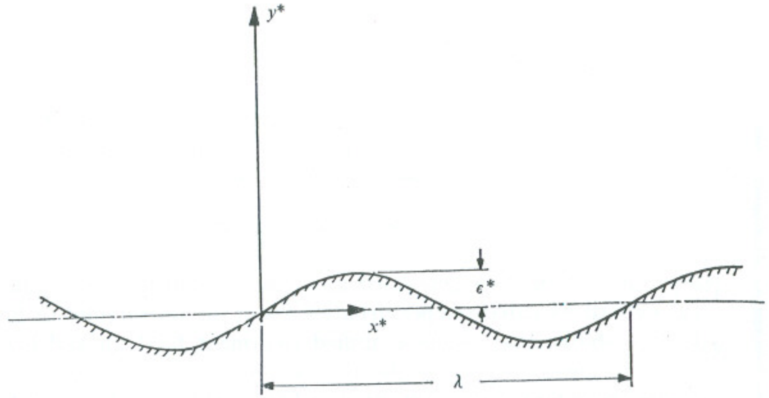

FIGURE 1. The wavy-wall problem. Incompressible fluid fills the region above the wall, which is described for $t^* < 0$ by $y^* = \epsilon^* \sin(2\pi x^*/\lambda)$. At $t^* = 0$ the wall is impulsively moved to the right with a velocity U.

Reynolds number limit in § 4, and in § 5 the solutions are discussed to bring out the special features of this problem.

2. Formulation

Consider incompressible fluid in a semi-infinite region bounded on one side by a wavy wall defined (see figure 1) for $t^* < 0$ by

$$y^* = \epsilon^* \sin(2\pi x^*/\lambda). \tag{1}$$

At time $t^* = 0$ the wall is impulsively given a velocity U in the x^* direction. In this two-dimensional situation the equations governing the motion of the incompressible fluid are

$$u^*_{x^*} + v^*_{y^*} = 0, \tag{2a}$$

$$u^*_{t^*} + u^* u^*_{x^*} + v^* u^*_{y^*} = \rho^{-1} p^*_{x^*} + \nu \nabla^2 u^*, \tag{2b}$$

$$v^*_{t^*} + u^* v^*_{x^*} + v^* v^*_{y^*} = \rho^{-1} p^*_{y^*} + \nu \nabla^2 v^*, \tag{2c}$$

where the symbols are defined in the usual way, the starred quantities are dimensional and the subscripts denote differentiation with respect to the subscript. The initial and boundary conditions are

$$u^*(x^*, y^*, t^*) = v(x^*, y^*, t^*) = 0 \quad \text{for} \quad t^* < 0, \tag{3a}$$

$$\left.\begin{array}{l} u^*(x^*, y^*, t^*) = U, \quad v^*(x^*, y^*, t^*) = 0 \quad \text{on} \\ y^* = \epsilon^* \sin\{2\pi\lambda^{-1}(x^* - Ut^*)\} \quad \text{for} \quad t^* \geqslant 0, \end{array}\right. \tag{3b}$$

$$u^*, v^* \to 0 \quad \text{as} \quad y^* \to \infty. \tag{3c}$$

We now define

$$\left.\begin{array}{lll} x = 2\pi x^*/\lambda, & u = u^*/U, & \epsilon = \epsilon^*/2\pi\lambda, \quad \text{etc.,} \\ t = 2\pi U t^*/\lambda, & p = p^*/\rho U^2, & R = U\lambda/2\pi\nu. \end{array}\right\} \tag{4}$$

In terms of these dimensionless variables and parameters the equations and boundary conditions for $t \geqslant 0$ take the form

$$u_x + v_y = 0, \tag{5a}$$

$$u_t + uu_x + vu_y = -p_x + R^{-1}\nabla^2 u, \tag{5b}$$

$$v_t + uv_x + vv_y = -p_y + R^{-1}\nabla^2 v, \tag{5c}$$

$$u(x,y,t) = 1, \quad v(x,y,t) = 0 \quad \text{on} \quad y = \epsilon \sin(x-t), \tag{6a}$$

$$u(x,y,t), v(x,y,t) \to 0 \quad \text{for} \quad y \to \infty. \tag{6b}$$

It proves convenient to use a non-orthogonal co-ordinate system in which one of the co-ordinates takes a constant value on the wall. We define (for $t \geqslant 0$)

$$\xi = x, \quad \eta = y - \epsilon \sin(x-t), \quad \tau = t, \tag{7a–c}$$

so that the wall is defined by $\eta = 0$. In terms of these co-ordinates (5a–c) become

$$u_\xi - \epsilon \cos(\xi - \tau) u_\eta + v_\eta = 0, \tag{8a}$$

$$\begin{aligned} &u_\tau + \epsilon \cos(\xi-\tau) u_\eta + u\{u_\xi - \epsilon\cos(\xi-\tau) u_\eta\} + vu_\eta \\ &\quad = -\{p_\xi - \epsilon \cos(\xi-\tau) p_\eta\} + R^{-1}\{u_{\xi\xi} - 2\epsilon\cos(\xi-\tau) u_{\xi\eta} \\ &\qquad + \epsilon \sin(\xi-\tau) u_\eta + \epsilon^2 \cos^2(\xi-\tau) u_{\eta\eta} + u_{\eta\eta}\}, \end{aligned} \tag{8b}$$

$$\begin{aligned} &v_\tau + \epsilon \cos(\xi-\tau) v_\eta + u\{v_\xi - \epsilon\cos(\xi-\tau) v_\eta\} + vv_\eta \\ &\quad = -p_\eta + R^{-1}\{v_{\xi\xi} - 2\epsilon\cos(\xi-\tau) v_{\xi\eta} + \epsilon\sin(\xi-\tau) v_\eta \\ &\qquad + \epsilon^2\cos^2(\xi-\tau) v_{\eta\eta} + v_{\eta\eta}\}. \end{aligned} \tag{8c}$$

The boundary conditions for $\tau \geqslant 0$ are

$$u(\xi,\eta,\tau) = 1, \quad v(\xi,\eta,\tau) = 0 \quad \text{on} \quad \eta = 0, \tag{9a}$$

$$u, v \to 0 \quad \text{as} \quad \eta \to \infty. \tag{9b}$$

Equations (8) are nonlinear and not amenable to solution as they stand. However, if the waviness of the wall is small it is natural to seek a perturbation solution for small ϵ. The limit $\epsilon = 0$ is, of course, the limit of a flat plate, for which the solution is well known. We therefore seek a perturbation solution about this unsteady solution. We assume expansions of the form

$$u = u^{(0)}(\eta,\tau) + \epsilon u^{(1)}(\xi,\eta,\tau) + O(\epsilon^2), \tag{10a}$$

$$v = \qquad \epsilon v^{(1)}(\xi,\eta,\tau) + O(\epsilon^2), \tag{10b}$$

$$p = p_0 \qquad + \epsilon p^{(1)}(\xi,\eta,\tau) + O(\epsilon^2). \tag{10c}$$

When these expansions are substituted into equations (8) and boundary conditions (9), equations to each order in ϵ may be obtained.

Equations to order 1

$$u^{(0)}_\tau = R^{-1} u^{(0)}_{\eta\eta}, \quad v^{(0)} = 0, \quad p^{(0)} = p_0, \tag{11}$$

$$u^{(0)}(0,\tau) = 1 \quad \text{for} \quad \tau \geqslant 0. \tag{12}$$

The solution is the well-known Rayleigh solution:

$$u^{(0)}(\eta,\tau) = 1 - \text{erf}(R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}) = \text{erfc}(R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}). \tag{13}$$

Equations to order ϵ. Eliminating the pressure by cross-differentiation, the equations for the velocity components are

$$u_\xi^{(1)}+v_\eta^{(1)} = \cos(\xi-\tau)\,u_\eta^{(0)}, \tag{14a}$$

$$\frac{\partial}{\partial\eta}[u_{\xi\xi}^{(1)}+u_{\eta\eta}^{(1)}-R\{u_\tau^{(1)}+u^{(0)}u_\xi^{(1)}+v^{(1)}u_\eta^{(0)}\}]-\frac{\partial}{\partial\xi}[v_{\xi\xi}^{(1)}+v_{\eta\eta}^{(1)}-R\{v_\tau^{(1)}+u^{(0)}v_\xi^{(1)}\}], \tag{14b}$$
$$=\frac{\partial}{\partial\eta}[-\sin(\xi-\tau)\,u_\eta^{(0)}+R\cos(\xi-\tau)\{u_\eta^{(0)}-u^{(0)}u_\eta^{(0)}\}].$$

The boundary conditions are

$$u^{(1)}=v^{(1)}=0 \quad \text{on} \quad \eta=0, \tag{15a}$$

$$u^{(1)}, v^{(1)}\to 0 \quad \text{as} \quad \eta\to\infty. \tag{15b}$$

Now (14a) and (14b) together with the boundary conditions (15a) and (15b) fully specify the problem for a wavy wall whose 'roughness' is small. This formulation is valid for all Reynolds numbers. Two facts are however immediately apparent. Since $u^{(0)}(\eta,\tau)$ is a function of the co-ordinates η and τ, equations (14) are a set of partial differential equations with non-constant coefficients. Thus even these linear equations are not easily amenable to solution. Second, it is now clear why the ξ, η, τ co-ordinate system is preferable to the x, y, t system. If the latter system is used the equations are simpler but the boundary conditions will lead to a formulation which will not be uniformly valid for small times.

In the next two sections we solve (14) in the low and high Reynolds number limits. The strategy is to solve the equations by reducing them to systems of equations with constant coefficients; only the leading terms in the Reynolds number expansions will be sought. The leading terms in these two limiting cases contain the gist of the physics of the problem.

3. Solution for small Reynolds number

When the fluid is very viscous we expect the Rayleigh limit to dominate the motion, i.e. that the fluid is dragged along by the plate, the vorticity being quickly diffused into the fluid. To a first approximation $u^{(0)}$ can be treated as uniform. Formally for $R\to 0$ we can write

$$\begin{aligned} u^{(0)}(\eta,\tau) &= 1-\mathrm{erf}\{R^{\frac12}\eta/2\tau^{\frac12}\} \\ &= 1-\left[\frac{2}{\pi^{\frac12}}\frac{R^{\frac12}\eta}{2\tau^{\frac12}}\left\{1-\frac{(\frac12 R^{\frac12}\eta/\tau^{\frac12})^2}{1\,!\,3}+\dots\right\}\right] \\ &= 1-\frac{1}{\pi^{\frac12}}\frac{R^{\frac12}\eta}{\tau^{\frac12}}+O(R^{\frac32}) \end{aligned} \tag{16}$$

and seek a solution for $u^{(1)}$, $v^{(1)}$ and $p^{(1)}$ of the form

$$u^{(1)}(\xi,\eta,\tau) = u_1^{(1)}(\xi,\eta,\tau)+u_2^{(1)}(\xi,\eta,\tau)+\dots, \tag{17a}$$

$$v^{(1)}(\xi,\eta,\tau) = v_1^{(1)}(\xi,\eta,\tau)+v_2^{(1)}(\xi,\eta,\tau)+\dots, \tag{17b}$$

$$p^{(1)}(\xi,\eta,\tau) = p_1^{(1)}(\xi,\eta,\tau)+p_2^{(1)}(\xi,\eta,\tau)+\dots, \tag{17c}$$

where $u_1^{(1)}$, etc., are the leading terms in an expansion for $R\to 0$ and successive terms $u_2^{(1)}$, etc., are the next terms in the expansion. Certain observations need to be made at this stage. The expansion in Reynolds number is actually contained in an expansion in the roughness parameter ϵ. Thus if higher-order terms were required $u^{(2)}$, $u^{(3)}$, etc., would be expanded in a similar fashion. Next, the expansion is not uniformly valid as it is clear that if $\eta/\tau^{\frac12}\gg R^{-\frac12}$ the second term in (16) is larger than the first term. Thus the expansion is valid for a layer initially close to the wall whose thickness increases like $\tau^{\frac12}$. The results show however that an outer expansion is not really necessary (to first order) as the perturbations decay at the wall and at infinity. Since we are mainly interested in the motion close to the wall the present inner expansion is sufficient. Third, we note that $u^{(0)}(\eta,\tau)$ is of exponential order in the Reynolds number; consequently derivatives of $u^{(0)}(\eta,\tau)$ and $u^{(1)}(\eta,\tau)$ contribute to different orders in a formal perturbation expansion. It is therefore best not to speculate in advance about the ordering in Reynolds number; it is in keeping with this strategy that (17) do not indicate the ordering in Reynolds number. The procedure followed is simply to keep all leading terms in $u^{(0)}$ and actually calculate the leading terms in (17). Thus we determine the ordering. Finally, only the leading terms $u_1^{(1)}(\xi,\eta,\tau)$, etc., will be calculated in this paper as they contain the kernel of the problem. In what follows we drop the subscript 1 on the understanding that $u^{(1)}(\xi,\eta,\tau)$, etc., really stand for $u_1^{(1)}(\xi,\eta,\tau)$, etc.

We take the forms indicated in (16) and (17) and substitute them into the governing equations (14) and boundary conditions (15). Retaining the leading terms we obtain

$$u_\xi^{(1)}+v_\eta^{(1)} = -\frac{R^{\frac12}\cos(\xi-\tau)}{(\pi\tau)^{\frac12}}\exp\left(-\frac{R\eta^2}{4\tau}\right), \tag{18a}$$

$$\frac{\partial}{\partial\eta}[u_{\xi\xi}^{(1)}+u_{\eta\eta}^{(1)}-R\{u_\tau^{(1)}+u_\xi^{(1)}\}]-\frac{\partial}{\partial\xi}[v_{\xi\xi}^{(1)}+v_{\eta\eta}^{(1)}-R\{v_\tau^{(1)}+v_\xi^{(1)}\}]$$
$$=-\sin(\xi-\tau)\frac{R^{\frac32}\eta\exp(-R\eta^2/4\tau)}{2(\pi\tau^3)^{\frac12}}+O(R^2\exp(-R\eta^2/4\tau)), \tag{18b}$$

$$u^{(1)}(\xi,0,\tau) = v^{(1)}(\xi,0,\tau) = 0, \tag{19a}$$

$$u^{(1)}, v^{(1)}\to 0 \quad \text{as} \quad \eta\to\infty. \tag{19b}$$

For this linear initial-value problem the Laplace transform method is appropriate. We define the transform and its inverse as

$$\tilde\phi(\xi,\eta,\sigma) = \int_0^\infty \phi(\xi,\eta,\tau)\,e^{-\sigma\tau}\,d\tau, \tag{20a}$$

$$\phi(\xi,\eta,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\tilde\phi(\xi,\eta,\sigma)\,e^{\sigma\tau}\,d\sigma. \tag{20b}$$

Equations (18) transform to

$$\tilde u_\xi^{(1)}+\tilde v_\eta^{(1)} = -\frac{R^{\frac12}}{2}\left[\frac{\exp\{-[R(\sigma-i)]^{\frac12}\eta-i\xi\}}{(\sigma-i)^{\frac12}}+\frac{\exp\{-[R(\sigma+i)]^{\frac12}\eta+i\xi\}}{(\sigma+i)^{\frac12}}\right], \tag{21a}$$

$$\frac{\partial}{\partial \eta}\{\tilde{u}^{(1)}_{\xi\xi}+\tilde{u}^{(1)}_{\eta\eta}-R\sigma\tilde{u}^{(1)}-R\tilde{u}^{(1)}_{\xi}\}-\frac{\partial}{\partial \xi}\{\tilde{v}^{(1)}_{\xi\xi}+\tilde{v}^{(1)}_{\eta\eta}-R\sigma\tilde{v}^{(1)}-R\tilde{v}^{(1)}_{\xi}\}$$
$$=\tfrac{1}{2}iR[-\exp\{-[R(\sigma-i)]^{\frac{1}{2}}\eta-i\xi\}+\exp\{-[R(\sigma+i)]^{\frac{1}{2}}\eta+i\xi\}]. \quad (21b)$$

The solutions to the homogeneous part of the above equations which decay as $\eta\to\infty$ may be written as

$$\tilde{u}^{(1)}_{H}=A\exp\{-\eta+i\xi\}+B\exp\{-\eta-i\xi\}$$
$$+C\exp\{-[1+R(\sigma-i)]^{\frac{1}{2}}\eta+i\xi\}+D\exp\{-[1+R(\sigma+i)]^{\frac{1}{2}}\eta-i\xi\}, \quad (22a)$$

$$\tilde{v}^{(1)}_{H}=iA\exp\{-\eta+i\xi\}-iB\exp\{-\eta-i\xi\}$$
$$+\frac{iC}{[1+R(\sigma-i)]^{\frac{1}{2}}}\exp\{-[1+R(\sigma-i)]^{\frac{1}{2}}\eta+i\xi\}-\frac{iD}{[1+R(\sigma+i)]^{\frac{1}{2}}}$$
$$\times\exp\{-[1+R(\sigma+i)]^{\frac{1}{2}}\eta-i\xi\}. \quad (22b)$$

The coefficients A, B, C and D are functions of σ and R alone. The particular solutions satisfying the inhomogeneous equations to leading order in R are

$$\tilde{u}^{(1)}_{P}=-\frac{iR^{\frac{1}{2}}}{2(\sigma-i)^{\frac{1}{2}}}\exp\{-[R(\sigma-i)]^{\frac{1}{2}}\eta-i\xi\}+\frac{iR^{\frac{1}{2}}}{2(\sigma+i)^{\frac{1}{2}}}\exp\{-[R(\sigma+i)]^{\frac{1}{2}}\eta+i\xi\}, \quad (23a)$$

$$\tilde{v}^{(1)}_{P}=O(R). \quad (23b)$$

The general solutions to (21) are thus

$$\tilde{u}^{(1)}=\tilde{u}^{(1)}_{P}+\tilde{u}^{(1)}_{H},\quad \tilde{v}^{(1)}=\tilde{v}^{(1)}_{P}+\tilde{v}^{(1)}_{H}. \quad (24a,b)$$

We now apply the boundary conditions (19) at $\eta=0$; the coefficients A, B, C and D are determined to leading order to be

$$A=-iR^{\frac{1}{2}}/2(\sigma+i)^{\frac{1}{2}}\{1-[1+R(\sigma-i)]^{\frac{1}{2}}\} \quad (25a)$$

$$B=iR^{\frac{1}{2}}/2(\sigma-i)^{\frac{1}{2}}\{1-[1+R(\sigma+i)]^{\frac{1}{2}}\}, \quad (25b)$$

$$C=-[1+R(\sigma-i)]^{\frac{1}{2}}A,\quad D=-[1+R(\sigma+i)]^{\frac{1}{2}}B. \quad (25c,d)$$

Thus for $R\to 0$ we have

$$u^{(1)}(\xi,\eta,\tau)=R^{\frac{1}{2}}\sin(\xi-\tau)[e^{-\eta}f_1(\tau)-f_2(\eta,\tau)-f_3(\eta,\tau)], \quad (26a)$$

$$v^{(1)}(\xi,\eta,\tau)=R^{\frac{1}{2}}\cos(\xi-\tau)[e^{-\eta}f_1(\tau)-f_4(\eta,\tau)], \quad (26b)$$

where
$$\bar{f}_1(\sigma)=1/\{\sigma^{\frac{1}{2}}[1-(1+R\sigma)^{\frac{1}{2}}]\}, \quad (27a)$$

$$\bar{f}_2(\eta,\sigma)=\{(1+R\sigma)^{\frac{1}{2}}\exp[-(1+R\sigma)^{\frac{1}{2}}\eta]\}/\{\sigma^{\frac{1}{2}}[1-(1+R\sigma)^{\frac{1}{2}}]\}, \quad (27b)$$

$$\bar{f}_3(\eta,\sigma)=\exp[-(R\sigma)^{\frac{1}{2}}\eta]/\sigma^{\frac{1}{2}}, \quad (27c)$$

$$\bar{f}_4(\eta,\sigma)=\exp[-(1+R\sigma)^{\frac{1}{2}}\eta]/\{\sigma^{\frac{1}{2}}[1-(1+R\sigma)^{\frac{1}{2}}]\}. \quad (27d)$$

The inversion of the transforms is elementary but tedious. The full solution to order ϵ in the limit $R\to 0$ is then given by

$$u(\xi,\eta,\tau)=\mathrm{erfc}\{R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}\}+\epsilon R^{\frac{1}{2}}\sin(\xi-\tau)$$
$$\times[e^{-\eta}f_1(\tau)-f_2(\eta,\tau)-f_3(\eta,\tau)]+O(\epsilon^2), \quad (28a)$$

$$v(\xi,\eta,\tau)=\epsilon R^{\frac{1}{2}}\cos(\xi-\tau)[e^{-\eta}f_1(\tau)-f_4(\eta,\tau)]+O(\epsilon^2), \quad (28b)$$

where the functions $f_1,\dots,f_4$ are defined by

$$f_1(\tau)=-\frac{1}{R}\left\{\frac{2\tau^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}+\frac{1}{\pi}\int_0^{1/R}\frac{e^{-r\tau}\{\tau(2-2rR)+R\}}{[r(1-rR)]^{\frac{1}{2}}}\,dr\right\}, \quad (29a)$$

$$f_2(\eta,\tau)=R^{-1}[\partial g_1/\partial\eta-\partial^2 g_1/\partial\eta^2], \quad (29b)$$

$$f_3(\eta,\tau)=(\pi\tau)^{-\frac{1}{2}}\exp(-R\eta^2/4\tau), \quad (29c)$$

$$f_4(\eta,\tau)=-R^{-1}[g_1-\partial g_1/\partial\eta] \quad (29d)$$

and where

$$g_1(\eta,\tau)=\frac{2}{\pi}\int_{1/R}^{\infty}\frac{e^{-r\tau}}{r^{\frac{1}{2}}}\left[\tau\cos[(rR-1)^{\frac{1}{2}}\eta]+\frac{R\eta}{2(rR-1)^{\frac{1}{2}}}\sin[(rR-1)^{\frac{1}{2}}\eta]\right]dr$$
$$+\frac{2}{\pi}\int_0^{1/R}\frac{\exp[-r\tau-(1-Rr)^{\frac{1}{2}}\eta]}{r^{\frac{1}{2}}}\left\{\tau-\frac{R\eta}{2(1-Rr)^{\frac{1}{2}}}\right\}dr. \quad (30)$$

The above solutions are well behaved for all (ξ,η,τ). Inspection of the solutions will vindicate the procedure used here of not specifying the forms of the expansions in advance. The dependence on the Reynolds number is quite different at small and large times. The formal solutions obtained above can be differentiated and be easily shown to satisfy the governing equations (5) and (6) to order ϵ and to leading order in R.

In order to show more clearly the nature of the solutions we now present the solutions valid for small and large times respectively. These can be obtained directly from (28)–(30) or more easily from the transforms using the theorems valid for small and large times.

Solution for small τ

$$u(\xi,\eta,\tau)=\mathrm{erfc}\{R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}\}+\epsilon\sin(\xi-\tau)[\mathrm{erfc}(R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}})$$
$$-\exp(-\eta)+O(\tau^{\frac{1}{2}})]+O(\epsilon^2), \quad (31a)$$

$$v(\xi,\eta,\tau)=\epsilon\cos(\xi-\tau)[\mathrm{erfc}(R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}})-\exp(-\eta)+O(\tau^{\frac{1}{2}})]+O(\epsilon^2). \quad (31b)$$

Solution for large τ

$$u(\xi,\eta,\tau)=\mathrm{erfc}\left\{\frac{R^{\frac{1}{2}}\eta}{2\tau^{\frac{1}{2}}}\right\}+\frac{\epsilon R^{\frac{1}{2}}\sin(\xi-\tau)}{(\pi\tau)^{\frac{1}{2}}}$$
$$\times[(1-\eta)e^{-\eta}-\exp(-R\eta^2/4\tau)]+\text{h.o.t.}, \quad (32a)$$

$$v(\xi,\eta,\tau)=-\epsilon R^{\frac{1}{2}}\cos(\xi-\tau)\,\eta e^{-\eta}/(\pi\tau)^{\frac{1}{2}}+\text{h.o.t.} \quad (32b)$$

The above solutions make clear the dangers of prescribing the ordering in Reynolds number in advance. The solutions can easily be seen to satisfy the governing equations (5) and the boundary conditions (6). These results will be discussed in greater detail in § 5.

4. The high Reynolds number limit

When the viscosity of the fluid is small the layer of fluid dragged along by the wall grows only slowly in time. The fluid field feels the waviness of the wall for some time before it is damped out by the wall boundary layer. We therefore expect the potential field which develops in the far field to be only slowly attenuated by the viscous field.

The zeroth-order velocity field is once again expressed as an asymptotic expansion, now however for $R\to\infty$:

$$\begin{aligned} u^{(0)}(\eta,\tau) &= 1-\operatorname{erf}(R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}) \\ &= 1-\left[1-\frac{\exp(-R\eta^2/4\tau)}{\pi^{\frac{1}{2}}\{R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}\}}\left\{1-\frac{2!}{1!\{R^{\frac{1}{2}}\eta/\tau^{\frac{1}{2}}\}^2}+\ldots\right\}\right] \\ &= \frac{2\tau^{\frac{1}{2}}}{(\pi R)^{\frac{1}{2}}\eta}\exp\left\{-\frac{R\eta^2}{4\tau}\right\}+O\left(R^{-\frac{3}{2}}\exp\left(-\frac{R\eta^2}{4\tau}\right)\right). \end{aligned} \tag{33}$$

We now seek a solution for $u^{(1)}$ and $v^{(1)}$ of the same form as (17) but with the expansion now for $R\to\infty$. Note that the expansion for $u^{(0)}$ is truly valid only for $\tau^{\frac{1}{2}} < R^{\frac{1}{2}}\eta$; naturally, the expansion is valid away from the wall outside a layer growing with time. Thus we expect even at this stage to have to construct a separate inner expansion to be matched to this outer expansion. As we shall see, this inner expansion turns out to be a remarkably simple boundary layer.

Once again, in what follows $u^{(1)}(\xi,\eta,\tau)$, etc., should be taken to mean $u_1^{(1)}(\xi,\eta,\tau)$, etc.; that is, the first terms in an asymptotic expansion for $R\to\infty$. The governing equations (15) now simplify to

$$u_\xi^{(1)}+v_\eta^{(1)} = -\frac{R^{\frac{1}{2}}}{(\pi\tau)^{\frac{1}{2}}}\cos(\xi-\tau)\exp\left\{-\frac{R\eta^2}{4\tau}\right\}, \tag{34a}$$

$$\frac{\partial}{\partial\eta}\{u_{\xi\xi}^{(1)}+u_{\eta\eta}^{(1)}-Ru_\tau^{(1)}\}-\frac{\partial}{\partial\xi}\{v_{\xi\xi}^{(1)}+v_{\eta\eta}^{(1)}-Rv_\tau^{(1)}\} = \frac{R^{\frac{5}{2}}\eta\cos(\xi-\tau)\exp\{-R\eta^2/4\tau\}}{2(\pi\tau^3)^{\frac{1}{2}}} + \text{h.o.t.}, \tag{34b}$$

In the above we have apparently inconsistently dropped the convection terms due to the normal velocity $v^{(1)}$; while this is done for pragmatic reasons, it can be easily justified on the basis of the rapid decay of $u^{(0)}(\eta,\tau)$ and by inspecting the resulting solutions. In the inner expansion however, these terms turn out to be dominant.

As in § 3 we Laplace transform (34):

$$\tilde{u}_\xi^{(1)}+\tilde{v}_\eta^{(1)} = -\frac{R^{\frac{1}{2}}}{2}\left[\frac{\exp\{-[R(\sigma-i)]^{\frac{1}{2}}\eta-i\xi\}}{(\sigma-i)^{\frac{1}{2}}}+\frac{\exp\{-[R(\sigma+i)]^{\frac{1}{2}}\eta+i\xi\}}{(\sigma+i)^{\frac{1}{2}}}\right], \tag{35a}$$

$$\frac{\partial}{\partial\eta}\{\tilde{u}_{\xi\xi}^{(1)}+\tilde{u}_{\eta\eta}^{(1)}-R\sigma\tilde{u}^{(1)}\}-\frac{\partial}{\partial\xi}\{\tilde{v}_{\xi\xi}^{(1)}+\tilde{v}_{\eta\eta}^{(1)}-R\sigma\tilde{v}^{(1)}\} = \tfrac{1}{2}R^2[\exp\{-[(\sigma-i)R]^{\frac{1}{2}}\eta-i\xi\}+\exp\{-[(\sigma+i)R]^{\frac{1}{2}}\eta+i\xi\}]. \tag{35b}$$

The solutions to the homogeneous equations are now

$$\tilde{u}_H^{(1)} = e^{-\eta}[A_0\cos\xi+B_0\sin\xi]+\exp[-(R\sigma)^{\frac{1}{2}}\eta][A_1\cos\xi+B_1\sin\xi], \tag{36a}$$

$$\tilde{v}_H^{(1)} = e^{-\eta}[-A_0\sin\xi+B_0\cos\xi]+(R\sigma)^{-\frac{1}{2}}\exp[-(R\sigma)^{\frac{1}{2}}\eta][A_1\sin\xi-B_1\cos\xi], \tag{36b}$$

where A_0, B_0, A_1 and B_1 are functions of σ and R alone. Particular integrals to the inhomogeneous equations are

$$\begin{aligned} \tilde{u}_P^{(1)} &= \frac{R^{\frac{1}{2}}}{2(\sigma-i)^{\frac{1}{2}}}\left[\frac{\cos\xi}{i}-\sin\xi\right]\exp(-[R(\sigma-i)]^{\frac{1}{2}}\eta) \\ &\quad -\frac{R^{\frac{1}{2}}}{2(\sigma+i)^{\frac{1}{2}}}\left[\frac{\cos\xi}{i}+\sin\xi\right]\exp(-[R(\sigma+i)]^{\frac{1}{2}}\eta), \end{aligned} \tag{37a}$$

$$\tilde{v}_P^{(1)} = O(1), \quad \text{i.e. of lower order.} \tag{37b}$$

Thus the complete solution is given by

$$\tilde{u}^{(1)} = \tilde{u}_H^{(1)}+\tilde{u}_P^{(1)}, \quad \tilde{v}^{(1)} = \tilde{v}_H^{(1)}+\tilde{v}_P^{(1)}. \tag{38a, b}$$

Now some care is required to obtain the solution. While the particular integrals (37) decay correctly at infinity, they grow as $R^{\frac{1}{2}}$ as $\eta\to 0$. In order that $u^{(1)}$ and $v^{(1)}$ match the inner solutions these terms need to be eliminated for $\eta\to 0$. This can easily be done if one notes that parts of the homogeneous solutions (36) are of exponential order in R; these can be used to eliminate the offending particular integrals for $\eta\to 0$. We therefore split up A_1 and B_1 into parts of order R and parts of order 1 as follows:

$$A_1 = A_1'+A_1'', \quad B_1 = B_1'+B_1'', \tag{39}$$

where

$$A_1' = -\left\{\frac{R^{\frac{1}{2}}}{2i(\sigma-i)^{\frac{1}{2}}}-\frac{R^{\frac{1}{2}}}{2i(\sigma+i)^{\frac{1}{2}}}\right\} = \frac{R^{\frac{1}{2}}}{2i}\frac{(\sigma+i)^{\frac{1}{2}}-(\sigma-i)^{\frac{1}{2}}}{(\sigma^2+1)^{\frac{1}{2}}}, \tag{40a}$$

$$B_1' = -\left\{-\frac{R^{\frac{1}{2}}}{2(\sigma-i)^{\frac{1}{2}}}-\frac{R^{\frac{1}{2}}}{2(\sigma+i)^{\frac{1}{2}}}\right\} = \frac{R^{\frac{1}{2}}}{2}\frac{(\sigma+i)^{\frac{1}{2}}+(\sigma-i)^{\frac{1}{2}}}{(\sigma^2+1)^{\frac{1}{2}}} \tag{40b}$$

and A_1'' and B_1'' are of order 1 and are to be determined by matching with the inner solution. The outer solution therefore takes the following form:

$$\begin{aligned} \tilde{u}^{(1)} &= e^{-\eta}[A_0\cos\xi+B_0\sin\xi]+\exp[-(R\sigma)^{\frac{1}{2}}\eta][A_1''\cos\xi+B_1''\sin\xi] \\ &\quad +R^{\frac{1}{2}}\left[\left\{\frac{\cos\xi}{i}-\sin\xi\right\}\frac{\exp\{-[R(\sigma-i)]^{\frac{1}{2}}\eta\}}{2(\sigma-i)^{\frac{1}{2}}}-\left\{\frac{\cos\xi}{i}+\sin\xi\right\}\frac{\exp\{-[R(\sigma+i)]^{\frac{1}{2}}\eta\}}{2(\sigma+i)^{\frac{1}{2}}}\right. \\ &\quad -\left\{\frac{\cos\xi}{2i(\sigma-i)^{\frac{1}{2}}}-\frac{\cos\xi}{2i(\sigma+i)^{\frac{1}{2}}}\right\}\exp[-(R\sigma)^{\frac{1}{2}}\eta]+\left\{\frac{\sin\xi}{2(\sigma-i)^{\frac{1}{2}}}+\frac{\sin\xi}{2(\sigma+i)^{\frac{1}{2}}}\right\} \\ &\quad \left.\times\exp[-(R\sigma)^{\frac{1}{2}}\eta]\right], \end{aligned} \tag{41a}$$

$$\begin{aligned} \tilde{v}^{(1)} &= e^{-\eta}[-A_0\sin\xi+B_0\cos\xi]-\frac{\exp[-(R\sigma)^{\frac{1}{2}}\eta]}{(R\sigma)^{\frac{1}{2}}}[A_1''\sin\xi-B_1''\cos\xi] \\ &\quad +\frac{\exp[-(R\sigma)^{\frac{1}{2}}\eta]}{\sigma^{\frac{1}{2}}}\left[\left\{\frac{\sin\xi}{2i(\sigma-i)^{\frac{1}{2}}}-\frac{\sin\xi}{2i(\sigma+i)^{\frac{1}{2}}}\right\}+\left\{\frac{\cos\xi}{2(\sigma-i)^{\frac{1}{2}}}+\frac{\cos\xi}{2(\sigma+i)^{\frac{1}{2}}}\right\}\right]. \end{aligned} \tag{41b}$$

It is to be noted that in using part of the homogeneous solution to cancel terms of order $\sqrt{R}$ in $u^{(1)}$ for $\eta\to 0$, we have not caused difficulties to appear in $v^{(1)}$. Terms of order $\sqrt{R}$ in $u_H^{(1)}$ correspond to terms of order 1 in $v_H^{(1)}$: this is the direct result of having terms of exponential order in the expansion. We now have to find the inner solution to which the outer solution has to be matched.

Actually this can be done by inspection but we shall demonstrate the formal approach in the interests of continuity in the treatment.

Inner solution

Close to the wall, however high the Reynolds number, $u^{(0)}(\eta,\tau)$ will be order 1. Also, close to the wall the normal convective terms will be important. The natural scaling for the problem for obvious reasons is

$$\eta' = R^{\frac{1}{2}}\eta, \quad \partial/\partial\eta = R^{\frac{1}{2}}\partial/\partial\eta'. \tag{42a, b}$$

In terms of the inner variable the governing equations (14) take the form

$$u^{(1)}_{\xi} + R^{\frac{1}{2}}v^{(1)}_{\eta'} = R^{\frac{1}{2}}\cos(\xi-\tau)\,u^{(0)}_{\eta'}, \tag{43a}$$

$$\begin{aligned} u^{(1)}_{\xi\xi} + Ru^{(1)}_{\eta'\eta'} - Ru^{(1)}_{\tau} - R^{\frac{3}{2}}v^{(1)}u^{(0)}_{\eta'} - Ru^{(0)}u^{(1)}_{\xi} \\ = Rp^{(1)}_{\xi} + R^{\frac{3}{2}}\cos(\xi-\tau)\,u^{(0)}_{\eta'}\{1-u^{(0)}\} - R^{\frac{1}{2}}\sin(\xi-\tau)\,u^{(0)}_{\eta'}, \end{aligned} \tag{43b}$$

$$v^{(1)}_{\xi\xi} + Rv^{(1)}_{\eta'\eta'} - Rv^{(1)}_{\tau} + Ru^{(0)}v^{(1)}_{\xi} = R^{\frac{3}{2}}p^{(1)}_{\eta'}. \tag{43c}$$

Note that we have not used separate symbols for the inner and outer expansions. The inner solution to leading order turns out to be trivially simple:

$$v^{(1)}(\xi,\eta',\tau) = \cos(\xi-\tau)\,(u^{(0)}-1), \tag{44a}$$

$$u^{(1)}(\xi,\eta',\tau) = O(R^{-\frac{1}{2}}), \tag{44b}$$

$$p^{(1)}(\xi,\eta',\tau) = \text{constant} + O(R^{-\frac{1}{2}}). \tag{44c}$$

This solution satisfies (43) to leading order, and satisfies the boundary conditions at the wall, namely $v^{(1)}(\eta'=0) = u^{(1)}(\eta'=0) = 0$. The pressure is constant throughout the boundary layer, which essentially acts as a source for the outer solution.

Matching and the complete solution

The inner solution determines the boundary conditions for the outer solution:

$$u^{(1)}_{\text{outer}}(\eta\to 0) = u^{(1)}_{\text{inner}}(\eta'\to\infty) = 0, \tag{45a}$$

$$v^{(1)}_{\text{outer}}(\eta\to 0) = v^{(1)}_{\text{inner}}(\eta'\to\infty) = -\cos(\xi-\tau). \tag{45b}$$

Applying these boundary conditions to the solutions (41) we obtain the outer solution

$$\begin{aligned} \tilde{u}^{(1)} = e^{-\eta}\Bigg[&\left\{\frac{1}{(\sigma^2+1)} + \frac{1}{\sigma^{\frac{1}{2}}}\left(\frac{1}{2i(\sigma-i)^{\frac{1}{2}}} - \frac{1}{2i(\sigma+i)^{\frac{1}{2}}}\right)\right\}\cos\xi \\ &+ \left\{-\frac{\sigma}{\sigma^2+1} - \frac{1}{\sigma^{\frac{1}{2}}}\left(\frac{1}{2(\sigma-i)^{\frac{1}{2}}} + \frac{1}{2(\sigma+i)^{\frac{1}{2}}}\right)\right\}\sin\xi\Bigg] \\ &+ \exp[-(R\sigma)^{\frac{1}{2}}\eta]\left[-\left\{\frac{1}{\sigma^2+1} + \frac{1}{\sigma^{\frac{1}{2}}}\left(\frac{1}{2i(\sigma-i)^{\frac{1}{2}}} - \frac{1}{2i(\sigma+i)^{\frac{1}{2}}}\right)\right\}\cos\xi\right. \\ &\left. + \left\{\frac{\sigma}{\sigma^2+1} + \frac{1}{\sigma^{\frac{1}{2}}}\left(\frac{1}{2(\sigma-i)^{\frac{1}{2}}} + \frac{1}{2(\sigma+i)^{\frac{1}{2}}}\right)\right\}\sin\xi\right] \\ &+ R^{\frac{1}{2}}\left[\left\{\frac{\cos\xi}{i} - \sin\xi\right\}\frac{\exp\{-[R(\sigma-i)]^{\frac{1}{2}}\eta\}}{2(\sigma-i)^{\frac{1}{2}}} - \left\{\frac{\cos\xi}{i} + \sin\xi\right\}\right. \\ &\times\frac{\exp\{-[R(\sigma+i)]^{\frac{1}{2}}\eta\}}{2(\sigma+i)^{\frac{1}{2}}} \end{aligned}$$

$$\begin{aligned} &\left. - \left\{\frac{\cos\xi}{2i(\sigma-i)^{\frac{1}{2}}} - \frac{\cos\xi}{2i(\sigma+i)^{\frac{1}{2}}}\right\}\exp[-(R\sigma)^{\frac{1}{2}}\eta] + \left\{\frac{\sin\xi}{2(\sigma-i)^{\frac{1}{2}}} + \frac{\sin\xi}{2(\sigma+i)^{\frac{1}{2}}}\right\}\right. \\ &\left.\times\exp[-(R\sigma)^{\frac{1}{2}}\eta]\right], \end{aligned} \tag{46a}$$

$$\begin{aligned} \tilde{v}^{(1)} = e^{-\eta}\Bigg[&-\left\{\frac{1}{\sigma^2+1} + \frac{1}{\sigma^{\frac{1}{2}}}\left(\frac{1}{2i(\sigma-i)^{\frac{1}{2}}} - \frac{1}{2i(\sigma+i)^{\frac{1}{2}}}\right)\right\}\sin\xi \\ &- \left\{\frac{\sigma}{\sigma^2+1} + \frac{1}{\sigma^{\frac{1}{2}}}\left(\frac{1}{2(\sigma-i)^{\frac{1}{2}}} + \frac{1}{2(\sigma+i)^{\frac{1}{2}}}\right)\right\}\cos\xi\Bigg] \\ &+ \frac{\exp[-(R\sigma)^{\frac{1}{2}}\eta]}{\sigma^{\frac{1}{2}}}\left[\left\{\frac{1}{(\sigma-i)^{\frac{1}{2}}} - \frac{1}{(\sigma+i)^{\frac{1}{2}}}\right\}\frac{\sin\xi}{2i} + \left\{\frac{1}{(\sigma-i)^{\frac{1}{2}}} + \frac{1}{(\sigma+i)^{\frac{1}{2}}}\right\}\frac{\cos\xi}{2}\right]. \end{aligned} \tag{46b}$$

Thus to order ϵ the complete outer solution for $R\to\infty$ is

$$\begin{aligned} u(\xi,\eta,\tau) = \operatorname{erfc}\{R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}\} + \epsilon\Bigg[&e^{-\eta}\{-\sin(\xi-\tau) + f_1(\eta,\tau)\cos\xi - f_2(\eta,\tau)\sin\xi\} \\ &+ \left\{\exp[-(\tfrac{1}{2}R)^{\frac{1}{2}}\eta]\sin[\xi-\tau+(\tfrac{1}{2}R)^{\frac{1}{2}}\eta] - \frac{1}{\pi}\int_0^\infty \frac{e^{-r\tau}\sin[(Rr)^{\frac{1}{2}}\eta]}{(r^2+1)^{\frac{1}{2}}}\right. \\ &\left.\times(\cos\xi + r\sin\xi)\,dr - \cos\xi f_3(\eta,\tau) + f_4(\eta,\tau)\sin\xi\right\} \\ &+ R^{\frac{1}{2}}\left\{\frac{-\sin(\xi-\tau)}{(\pi\tau)^{\frac{1}{2}}}\exp\left(-\frac{R\eta^2}{4\tau}\right) - f_5(\eta,\tau)\cos\xi + f_6(\eta,\tau)\sin\xi\right\}\Bigg] \\ &+ O(\epsilon^2), \end{aligned} \tag{47a}$$

$$\begin{aligned} v(\xi,\eta,\tau) = \epsilon[&e^{-\eta}\{-\cos(\xi-\tau) - f_1(\eta,\tau)\sin\xi - f_2(\eta,\tau)\cos\xi\} \\ &+ \{f_3(\eta,\tau)\sin\xi + f_4(\eta,\tau)\cos\ \xi\}] + O(\epsilon^2), \end{aligned} \tag{47b}$$

where

$$f_1(\eta,\tau) = -\frac{1}{\pi}\int_0^\infty \frac{e^{-r\tau}}{r^{\frac{1}{2}}}\,\frac{a + a\cos\tau - b\sin\tau}{a^2+b^2}\,dr, \tag{48a}$$

$$f_2(\eta,\tau) = \frac{1}{\pi}\int_0^\infty \frac{e^{-r\tau}}{r^{\frac{1}{2}}}\,\frac{-b + b\cos\tau + a\sin\tau}{a^2+b^2}\,dr, \tag{48b}$$

$$f_3(\eta,\tau) = -\frac{1}{\pi}\int_0^\infty \frac{e^{-r\tau}}{r^{\frac{1}{2}}}\left[\frac{b\sin[\eta(Rr)^{\frac{1}{2}}] + \exp(-b\eta R^{\frac{1}{2}})\{a\cos(\tau-a\eta R^{\frac{1}{2}}) - b\sin(\tau-a\eta R^{\frac{1}{2}})\}}{a^2+b^2}\right]dr, \tag{48c}$$

$$f_4(\eta,\tau) = \frac{1}{\pi}\int_0^\infty \frac{e^{-r\tau}}{r^{\frac{1}{2}}}\left[\frac{a\sin[\eta(Rr)^{\frac{1}{2}}] + \exp(-b\eta R^{\frac{1}{2}})\{b\cos(\tau-a\eta R^{\frac{1}{2}}) + a\sin(\tau-a\eta R^{\frac{1}{2}})\}}{a^2+b^2}\right]dr, \tag{48d}$$

$$f_5(\eta,\tau) = -\frac{1}{\pi}\int_0^\infty e^{-r\tau}\left[\frac{b\sin[\eta(Rr)^{\frac{1}{2}}] + \exp(-b\eta R^{\frac{1}{2}})\{a\cos(\tau-a\eta R^{\frac{1}{2}}) - b\sin(\tau-a\eta R^{\frac{1}{2}})\}}{a^2+b^2}\right]dr, \tag{48e}$$

$$f_6(\eta,\tau) = \frac{1}{\pi}\int_0^\infty e^{-r\tau}\left[\frac{a\sin[\eta(Rr)^{\frac{1}{2}}] + \exp(-b\eta R^{\frac{1}{2}})\{b\cos(\tau-a\eta R^{\frac{1}{2}}) + a\sin(\tau-a\eta R^{\frac{1}{2}})\}}{a^2+b^2}\right]dr, \tag{48f}$$

$$a(r) = \{\tfrac{1}{2}[r+(r^2+1)^{\frac{1}{2}}]\}^{\frac{1}{2}}, \quad b(r) = 1/2a = \{2[r+(r^2+1)^{\frac{1}{2}}]\}^{-\frac{1}{2}}. \tag{48g}$$

The above outer solution satisfies the governing equations (14) and boundary conditions (15b) and (45) to leading order in the Reynolds number as $R \to \infty$. Finally we present the solutions valid for small and large times.

Small time solution. For $\tau \to 0$ the leading terms are

$$u(\xi,\eta,\tau) \sim \operatorname{erfc}\{R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}\} + \epsilon[2\sin\xi\{\operatorname{erfc}(R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}) - e^{-\eta}\} + \tfrac{1}{2}\eta\cos\xi\operatorname{erfc}\{R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}\}] + O(\epsilon^2), \quad (49a)$$

$$v(\xi,\eta,\tau) \sim \epsilon[\cos\xi\{\operatorname{erfc}(R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}} - 2e^{-\eta}\}] + O(\epsilon^2). \quad (49b)$$

Large time solution. For large times the leading terms are

$$u(\xi,\eta,\tau) \sim \operatorname{erfc}\{R^{\frac{1}{2}}\eta/2\tau^{\frac{1}{2}}\} + \epsilon[-e^{-\eta}\sin(\xi-\tau) + \exp\{-(\tfrac{1}{2}R)^{\frac{1}{2}}\eta\}\sin\{\xi-\tau+(\tfrac{1}{2}R)^{\frac{1}{2}}\eta\}] + O(\epsilon^2), \quad (50a)$$

$$v(\xi,\eta,\tau) \sim \epsilon[-e^{-\eta}\cos(\xi-\tau)] + O(\epsilon^2). \quad (50b)$$

It is evident that these solutions satisfy the boundary conditions and the equations to leading order in all parameters.

5. Discussion

In the previous three sections we have shown how one can obtain approximate solutions for the flow generated by the impulsive motion of an infinite wavy wall in a viscous fluid. The perturbation treatment, valid for waviness of small amplitude, was possible because the Rayleigh solution for the motion of a flat plate (i.e. one with no waviness) is known in closed form. By perturbing about this limiting solution one can obtain the solution for a wavy wall. The problem is still complicated however, because the Rayleigh solution is a function of space and time; this leads to partial differential equations with non-constant coefficients for the wavy-wall flow field. What we have shown here is that in the limiting cases $R \to 0$ and $R \to \infty$ the perturbation field can be treated by further asymptotic expansions of the wavy-wall field for small and large Reynolds numbers respectively. The two limiting cases show significantly different behaviour.

The flow field for $R \to 0$ is displayed in (28)–(32). In this limit the flow field is dominated by viscous effects. The wall drags the fluid and very quickly the waviness of the wall is essentially submerged in a layer of fluid moving parallel to the mean wall profile. For small times [see (31)] the velocity field far from the wall does have the classic potential behaviour, i.e.

$$u^{(1)} \sim -\epsilon\sin(\xi-\tau)\exp(-\eta), \quad v^{(1)} \sim -\epsilon\cos(\xi-\tau)\exp(-\eta).$$

However, as a result of the high viscosity, the viscous Rayleigh layer quickly erodes the potential field and the solution for large times emerges. For large times [see (32)] the zeroth-order field dominates the flow as the perturbations due to the wall waviness decay as $\tau^{-\frac{1}{2}}$.

At high Reynolds numbers, on the other hand, the picture is quite different. This limit is described by (47)–(50). The Rayleigh solution is of order one only in a layer very close to the wall; outside this layer the effects of viscosity decay exponentially. As the large time solutions [equations (50)] clearly indicate, the field away from the wall decays rapidly to the classical inviscid field.

The high Reynolds number solution has a number of interesting features. It is to be noted that the solution was derived in a somewhat unusual fashion. Normally, in standard boundary-layer problems one starts with the inviscid outer solution and then calculates the boundary layer. In the present calculations the inviscid part of the solution emerges naturally from the complete solution; in fact, this result boosts ones confidence in the present calculational procedure. The boundary-layer part of the perturbation solution [equation (44)] is remarkably simple and acts as a source as far as the outer solution is concerned. Actually the Rayleigh part of the complete solution contributes to the full boundary layer. Thus we have in effect computed the boundary layer in two stages, one to order one and one to order ϵ. As for the inviscid field that emerges, this is not (unlike the $R \to 0$ limit) attenuated by time directly; it is merely eroded by the slowly growing viscous boundary layer.

As we have pointed out earlier the whole calculation shows how carefully terms of exponential order have to be handled. Such terms on differentiation contribute to different orders of a formal perturbation expansion. The method used here was to retain all leading terms in the equations and actually calculate the leading terms of the solutions. If this simple procedure is not used one is forced to use the more formal method of matched asymptotic expansions; this will in general necessitate obtaining different expansions valid in different regions of ξ, η, τ space and matching them. We prefer the present simple approach at least for obtaining the leading terms.

We conclude with a few remarks, admittedly speculative, on the relevance of the present results to the problem of the generation of turbulence. If one examines the large time solution for $R \to \infty$, one notices that in a frame moving with the wavy wall there is no real unsteadiness in the solution except for the growth of the boundary layer. The present results therefore do not show any turbulence-like structure. This is not surprising as in two dimensions the vortex lines are straight and cannot be stretched or twisted. This conclusion appears to be supported by the mathematical work of Ladyzhenskaya (1959, 1969). Strictly speaking Ladyzhenskaya's results apply to two-dimensional fields in which the velocity is essentially square integrable over the whole field; this condition is not satisfied by the velocity field considered here. Intuitively it seems plausible that the violation of this single condition owing to the infinite extent in the x direction is unlikely to invalidate Ladyzhenskaya's result that turbulence is not possible in a two-dimensional field. Now the last few comments here are of a purely speculative nature based on our limited understanding of a difficult problem; the fact is, however, that the present two-dimensional calculation shows no turbulence-like structure. Whether in three dimensions a prototype problem similar to the present one will show the generation of turbulence is an open question.

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